

HILBERT MODULES OVER A PLANAR ALGEBRA AND THE HAAGERUP PROPERTY

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ABSTRACT. Given a subfactor planar algebra \mathcal{P} and a Hilbert \mathcal{P} -module of lowest weight 0 we build a bimodule over the symmetric enveloping inclusion associated to \mathcal{P} . As an application we prove diagrammatically that the Temperley-Lieb-Jones standard invariants have the Haagerup property. This provides a new proof of a result due to Popa and Vaes.

1. INTRODUCTION AND MAIN RESULTS

Popa initiated the study of approximation properties of subfactors in [Pop86, Pop94a, Pop94b]. To any finite index subfactor of type II_1 one can associate a combinatorial object called the standard invariant. This invariant has been axiomatized as a paragroup, a λ -lattice, and a planar algebra respectively by Ocneanu, Popa, and the second author [Ocn88, Pop95, Jonb]. An analogue of quantum doubles for a subfactor was introduced by Ocneanu, Longo and Rehren, and Popa ([Ocn88, LR95, Pop94b]). The latter construction is called the symmetric enveloping inclusion. For the construction of subfactors of Guionnet et al. in [GJS10], Curran et al. in [CJS14] gave a diagrammatic description of the symmetric enveloping inclusion.

Recently, Popa and Vaes introduced a representation theory for subfactors and standard invariants [PV]. They defined the Haagerup property for a subfactor and showed that it depends only on its standard invariant. They then showed that the Temperley-Lieb-Jones standard invariants have the Haagerup property. (Note, this result was already announced in [Pop06, Remark 3.5.5].) Their proof uses previous work on discrete quantum groups and the equivalence between the bimodule category associated to the Temperley-Lieb-Jones standard invariant and the representation category of the quantum group $\text{PSU}_q(2)$ [DCFY14]. Here we give another proof:

Theorem 1.1. *The Temperley-Lieb-Jones standard invariant has the Haagerup property for any loop parameter $\delta \in \{2 \cos \frac{\pi}{n}, n \geq 3\} \cup [2 : \infty)$.*

Our proof only uses planar algebra technology. The idea is that the lowest weight zero annular representations of the planar algebra immediately give "compact" bimodules (which are obvious in the Curran et al. pictures), which tend to the trivial bimodule in the way required by the Popa-Vaes definition of the Haagerup property.

Acknowledgement. The first author thanks Dietmar Bisch for many encouragements. He also thanks Jesse Peterson and Jean-Louis Lhuillier for their patience and guidance.

2. PRELIMINARIES

2.1. The symmetric enveloping inclusion associated to a subfactor planar algebra. We refer to [Jonb] for more details about planar algebras. We recall the construction of [CJS14, Section 2]. Note, we define the symmetric enveloping inclusion via the product introduced in [CJS14, Section 2.1] that we call the Bacher product. Let $\mathcal{P} = (\mathcal{P}_n^\pm, n \geq 0)$ be a subfactor

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planar algebra. For any $k, n, m \geq 0$, let $D_k(n, m)$ be a copy of the vector space \mathcal{P}_{n+m+2k}^+ . We decorate strings with natural numbers to indicate that they represent a given number of parallel strings. The distinguished interval of a box is decorated by a dollar sign if it is not at the top left corner. We will omit unnecessary decorations. Consider the direct sum

$$Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P} := \bigoplus_{n, m \geq 0} D_k(n, m)$$

that we equipped with the Bacher product:

$$x \star_k y = \sum_{a=0}^{\min(2n, 2i)} \sum_{b=0}^{\min(2m, 2j)} \text{Diagram}$$

where $x \in D_k(n, m)$ and $y \in D_k(i, j)$. Let $\dagger : Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P} \rightarrow Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$ be the anti-linear involution that sends $D_k(n, m)$ to itself and satisfies

$$x^\dagger = \text{Diagram}, \text{ for any } x \in D_k(n, m).$$

Consider the linear form $\tau : Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P} \rightarrow \mathbb{C}$, which is zero unless $n = m = 0$ and sends the unit of $D_k(0, 0)$ to 1. We have that $(Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}, \star_k, \dagger, \tau)$ is an associative $*$ -algebra with a faithful tracial state [CJS14, Corollary 2.3]. Further, $Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$ acts by bounded operators on the Gelfand-Naimark-Segal Hilbert space for τ [CJS14, Theorem 2.1]. Let $M_k \boxtimes M_k$ be its Gelfand-Naimark-Segal completion which is a factor of type II_1 . Let M_k be the von Neumann subalgebra of $M_k \boxtimes M_k$ generated by elements of the form

$$\text{Diagram} \in D_k(n, 0), \quad n \geq 0.$$

Observe, the von Neumann subalgebra of $M_k \boxtimes M_k$ generated by the family of sets $D_k(0, m)$, $m \geq 0$ is isomorphic to M_k^{op} and commutes with M_k . We identify M_k^{op} with this von Neumann subalgebra. Note, we have a unital inclusion of M_{k-1} in M_k by adding two horizontal strings under elements of M_{k-1} . By [GJS10], $M_{k-1} \subset M_k$ is a subfactor of type II_1 with standard invariant isomorphic to the subfactor planar algebra \mathcal{P} or its opposite depending on the parity of k . The von Neumann subalgebra of $M_k \boxtimes M_k$ generated by M_k and M_k^{op} is isomorphic to $M_k \overline{\otimes} M_k^{\text{op}}$ and the inclusion

$$M_k \vee M_k^{\text{op}} \subset M_k \boxtimes M_k$$

is isomorphic to Popa's symmetric enveloping inclusion associated to the subfactor $M_{k-1} \subset M_k$ for any $k \geq 1$ [CJS14]. We denote by $M \overline{\otimes} M^{\text{op}} \subset M \boxtimes M$ the inclusion $M_0 \vee M_0^{\text{op}} \subset M_0 \boxtimes M_0$ and call it the symmetric enveloping inclusion associated to \mathcal{P} . Similarly, we write $D_0(n, m) = D(n, m)$ for any $n, m \geq 0$.

2.2. Hilbert modules over a subfactor planar algebra. We introduce notations and terminology regarding Hilbert modules over a subfactor planar algebra. We refer to [Jona] for more details. Let us fix a subfactor planar algebra \mathcal{P} . An annular tangle α is a tangle in \mathcal{P} with the choice of a distinguished internal disc. We write $\text{Ann}\mathcal{P}((m, \varepsilon), (n, \epsilon))$ the complex vector space spanned by annular tangles with $2n$ (resp. $2m$) boundary points on its internal (resp. external) disc and

where the dollar sign is in a region with shading ϵ (resp. ε). A tangle in $\text{Ann}\mathcal{P}((m, \varepsilon), (n, \epsilon))$ is called a $((m, \varepsilon), (n, \epsilon))$ -annular tangle. Let $A\mathcal{P} = (A\mathcal{P}((m, \varepsilon), (n, \epsilon)), n, m \geq 0, \epsilon, \varepsilon \in \pm)$ be the annular algebroid associated to \mathcal{P} . We denote by $\alpha \mapsto \alpha^\dagger$ the anti-linear involution which sends a $((m, \varepsilon), (n, \epsilon))$ -annular tangle to a $((n, \epsilon), (m, \varepsilon))$ -annular tangle by reflection in a circle half way between the inner and outer boundaries. A Hilbert \mathcal{P} -module is a graded vector space $V = (V_n^\pm, n \geq 0)$, where each V_n^\pm is a finite dimensional Hilbert space, $A\mathcal{P}$ acts on V , and the inner product is compatible with this action. It means that if $\alpha \in A\mathcal{P}((m, \varepsilon), (n, \epsilon))$ then it defines a linear map from V_n^ϵ to V_m^ε such that

$$\langle \alpha(v), w \rangle = \langle v, \alpha^\dagger(w) \rangle, \text{ for any } v \in V_n^\epsilon, w \in V_m^\varepsilon.$$

The lowest weight of a Hilbert \mathcal{P} -module V is the smallest natural number n such that $V_n^+ \neq \{0\}$.

2.3. Hilbert TLJ -modules of lowest weight 0. Consider the Temperley-Lieb-Jones planar algebra $\mathcal{P} = TLJ$ with loop parameter $\delta \geq 2$.

Irreducible Hilbert TLJ -modules of lowest weight 0 have been fully classified in [Jona] and in [GL98] for the unshaded case. For any $0 < t \leq \delta$ there exists a Hilbert TLJ -module $V(t) = (V(t)_n^\pm, n \geq 0)$ such that $V(t)_0^+$ is one dimensional and spanned by a unit vector $\xi(t)$ which satisfies

$$\langle \alpha(\xi(t)), \beta(\xi(t)) \rangle = \delta^c t^{2d},$$

where α, β are annular tangles, c is the number of contractible circles in the (\pm, \pm) -annular tangle $\beta^\dagger \circ \alpha$ and d is half the number of non-contractible ones. Those Hilbert TLJ -modules will be used to construct unital completely positive maps on the symmetric enveloping inclusion associated to the Temperley-Lieb-Jones planar algebra.

3. HILBERT \mathcal{P} -MODULES GIVE $(M \overline{\otimes} M^{\text{op}} \subset M \boxtimes M)$ -BIMODULES

Let $V = (V_n^\pm, n \geq 0)$ be a Hilbert \mathcal{P} -module of lowest weight 0. For $i, j \geq 0$, let $\mathcal{H}_{i,j}$ be a copy of the Hilbert space V_{i+j}^+ . Let $\mathcal{H} = \bigoplus_{i,j \geq 0} \mathcal{H}_{i,j}$ be the Hilbert space equal to the direct sum of the $\mathcal{H}_{i,j}$. In particular, $\mathcal{H}_{i+1,j-1}$ is orthogonal to $\mathcal{H}_{i,j}$ in \mathcal{H} . Consider the dense pre-Hilbert subspace $\mathcal{K} \subset \mathcal{H}$ spanned by the union of all $\mathcal{H}_{i,j}$. We put

$$\pi_0(x)\xi = \sum_{a,b} \left(\begin{array}{c} \text{Diagram: A rectangle containing two boxes labeled } x \text{ and } \xi. A circle with label } a \text{ is above the boxes, and a circle with label } b \text{ is below them.} \end{array} \right),$$

for any $x \in D(n, m) \subset Gr\mathcal{P} \boxtimes Gr\mathcal{P}$ and $\xi \in \mathcal{H}_{i,j}$. This defines a representation

$$\pi_0 : Gr\mathcal{P} \boxtimes Gr\mathcal{P} \longrightarrow \mathcal{L}(\mathcal{K}),$$

where $\mathcal{L}(\mathcal{K})$ is the algebra of endomorphism of the vector space \mathcal{K} .

Proposition 3.1. *For any $x \in Gr\mathcal{P} \boxtimes Gr\mathcal{P}$, $\pi_0(x)$ defines a bounded operator on \mathcal{H} . Further, the representation π_0 extends to a normal $*$ -representation*

$$\pi : M \boxtimes M \longrightarrow \mathcal{B}(\mathcal{H}).$$

Proof. Consider x in $Gr\mathcal{P} \boxtimes Gr\mathcal{P}$. We can prove that $\pi_0(x)$ defines a bounded operator by following a similar argument than [JSW10, Theorem 3.3]. We continue to denote by $\pi_0(x)$ its extension to \mathcal{H} . Let $\xi \in \mathcal{H}_{0,0}$ be a unit vector and let ω_ξ be its associated vector state. Note, $\omega_\xi \circ \pi_0(x) = \tau(x)$ for any $x \in Gr\mathcal{P} \boxtimes Gr\mathcal{P}$, where τ is the unique normal tracial state on $M \boxtimes M$. Therefore, π_0 extends to a normal $*$ -representation $\pi : M \boxtimes M \longrightarrow \mathcal{B}(\mathcal{H})$. \square

Recall, if $T \subset S$ is an inclusion of von Neumann algebras, then a Hilbert $(T \subset S)$ -module is a couple (\mathcal{H}, ξ) such that \mathcal{H} is a Hilbert S -module and ξ is a T -central vector of \mathcal{H} .

Proof. Proposition 3.1 implies that \mathcal{H} is a $M \boxtimes M$ -bimodule with the action described above. Consider $x \otimes y^{\text{op}} \in Gr\mathcal{P} \otimes Gr\mathcal{P}^{\text{op}}$, where $Gr\mathcal{P} \otimes Gr\mathcal{P}^{\text{op}} = M \overline{\otimes} M^{\text{op}} \cap Gr\mathcal{P} \boxtimes Gr\mathcal{P}$. Since $\xi \in \mathcal{H}_{0,0}$, we have

Diagrammatic representation of the associativity property for the multiplication in the tensor product of two algebras. The left side shows the expression $(x \otimes y^{\text{op}}) \cdot \xi$, where x and y are in a box, and ξ is outside. The right side shows $\xi \cdot (x \otimes y^{\text{op}})$, where ξ is in a box, and x and y are in a box. The two sides are connected by an equals sign.

4. THE TEMPERLEY-LIEB-JONES STANDARD INVARIANT HAS THE HAAGERUP PROPERTY

Definition 4.1. Consider an inclusion of tracial von Neumann algebras $\mathcal{N} \subset (\mathcal{M}, \tau)$. A completely positive approximation of the identity (CPAI) for $\mathcal{N} \subset (\mathcal{M}, \tau)$ is a sequence of normal \mathcal{N} -bimodular trace-preserving unital completely positive maps $(\varphi_l : \mathcal{M} \rightarrow \mathcal{M}, l \geq 0)$ such that $\|\varphi_l(x) - x\|_2 \rightarrow_l 0$, for any $x \in \mathcal{M}$, and the unique continuous extension $\Theta_l \in B(L^2(\mathcal{M}, \tau))$ of φ_l to $L^2(\mathcal{M}, \tau)$ is in the compact ideal space of $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$.

Definition 4.2. [PV] A subfactor $N \subset M$ has the Haagerup property if its symmetric enveloping inclusion has the relative Haagerup property. A standard invariant \mathcal{G} has the Haagerup property if there exists a subfactor $N \subset M$ with standard invariant isomorphic to \mathcal{G} which has the Haagerup property.

Lemma 4.3. *Let \mathcal{P} be a subfactor planar algebra. Then \mathcal{P} has the Haagerup property if and only if its associated symmetric enveloping inclusion $M \overline{\otimes} M^{op} \subset M \boxtimes M$ has the relative Haagerup property.*

M_1 . Consider the inclusion $M_0 \vee M_0^{\text{op}} \subset M_1 \boxtimes M_1$. Let e be the Jones projection $e = \frac{1}{\delta} \begin{array}{c} \circlearrowleft \\ | \\ \circlearrowright \end{array}$.

Note, the compression $e(M_0 \vee M_0^{\text{op}})e \subset e(M_1 \boxtimes M_1)e$ is isomorphic to $M \overline{\otimes} M^{\text{op}} \subset \overline{M \boxtimes M}$. Therefore, by [Pop06, Proposition 2.3 and Proposition 2.4], $M \overline{\otimes} M^{\text{op}} \subset M \boxtimes M$ has the relative Haagerup property if and only if $M_1 \vee M_1^{\text{op}} \subset M_1 \boxtimes M_1$ has the relative Haagerup property. \square

Lemma 4.4. *Let TLJ be the Temperley-Lieb-Jones planar algebra with a loop parameter $\delta \geq 2$ and let $M \overline{\otimes} M^{\text{op}} \subset M \boxtimes M$ be its associated symmetric enveloping inclusion. Consider the $2n$ th-Jones-Wenzl idempotent $g_n \in TLJ_{2n}^+$ that we identify with its associated element in $D(n, n) \subset M \boxtimes M$. Let $L_n \subset L^2(M \boxtimes M)$ be the $M \overline{\otimes} M^{\text{op}}$ -bimodule generated by g_n . Then L_n is isomorphic to $X_n \overline{\otimes} \overline{X_n^{\text{op}}}$, where X_n is the irreducible M_0 -bimodule corresponding to the $2n$ th vertex in the principal graph of the subfactor $M_0 \subset M_1$. Further, $L^2(M \boxtimes M)$ is equal to the direct sum of the bimodules L_n .*

Proof. We follow an argument in [CJS14, pp. 120-122]. Let us show that L_n is orthogonal to L_m if $n \neq m$. This is equivalent to show that for any $x, y \in TLJ$, we have $xg_ny \perp g_m$ in the planar algebra TLJ . But this is obvious. Observe, the $*$ -algebra $Gr\mathcal{P} \boxtimes Gr\mathcal{P}$ is generated by the set of Jones-Wenzl idempotents and $GrP \otimes GrP^{\text{op}}$. Therefore, $L^2(M \boxtimes M)$ is equal to the direct sum of the bimodules L_n . Consider the M -bimodule $X_n \subset L^2(M_n)$ equal to the image of g_n viewed as an element of $TLJ_{2n}^+ = M' \cap M_{2n} \subset B(L^2(M_n))$. We have an isomorphism from $X_n \overline{\otimes} \overline{X_n^{\text{op}}}$ onto L_n given by the tangle which connects the $2n$ side strings of an element of X_n (resp. $\overline{X_n^{\text{op}}}$) to the top strings of g_n (resp. the bottom strings of g_n). \square

Theorem 4.5. *Let TLJ be the Temperley-Lieb-Jones planar algebra with any loop parameter $\delta \in \{2 \cos(\frac{\pi}{n}), n \geq 3\} \cup [2 : \infty)$. Then TLJ has the Haagerup property.*

Proof. If $\delta = 2 \cos(\frac{\pi}{n})$ for some $n \geq 3$, then TLJ has finite depth. Therefore, its symmetric enveloping inclusion is a subfactor of finite index. This implies that TLJ has the Haagerup property. We assume that $\delta \geq 2$. We write $T = M \overline{\otimes} M^{\text{op}}$ and $S = M \boxtimes M$. Consider $0 < t < \delta$ and the pointed Hilbert TLJ -module $(V(t), \xi(t))$ of section 2.3 where $\xi(t) \in V(t)_0^+$ is a unit vector. Let (H^t, ξ^t) be its associated $(T \subset S)$ -bimodule as constructed in section 3. Let $Z_t : L^2(S) \rightarrow H^t$ be the continuous linear map densely defined as follows $Z_t(x\Omega) = \xi^t \cdot x$, for any $x \in S$. Define the normal T -bimodular unital completely positive map $\phi_t : S \rightarrow S$ by the formula $\phi_t(x) = Z_t^* \pi_t(x) Z_t$, where $\pi_t : S \rightarrow B(H^t)$ is the left action of S on H^t . We will show that the net $(\phi_t, 0 < t < \delta)$ is the desired approximation of the identity.

Note, the T -bimodules L_n are isomorphic to $X_n \overline{\otimes} \overline{X_n^{\text{op}}}$ for any $n \geq 0$. Hence, they are irreducible and pairwise non-isomorphic. By Schur's Lemma, there exists a scalar valued function $c_t : \mathbb{N} \rightarrow \mathbb{C}$ such that $\Theta_t = \sum_{n \geq 0} c_t(n) s_n$, where Θ_t is the unique continuous extension of ϕ_t to $L^2(S)$ and s_n is the orthogonal projection from $L^2(S)$ onto L_n . We have the formula

$$c_t(n) = \frac{\langle \phi_t(g_n), g_n \rangle}{\langle g_n, g_n \rangle}, \text{ for any } n \geq 0.$$

Let τ_{2n} be the non-normalized trace of the C^* -algebra TLJ_{2n}^+ . Remark, $\tau_{2n}(g_n) = \langle g_n, g_n \rangle$, for any $n \geq 0$. Let q be the unique real number bigger than 1 satisfying $q + q^{-1} = \delta$. It is well known that $\tau_{2n}(g_n) = [2n + 1]_q$, where

$$[2n + 1]_q = \frac{q^{2n+1} - q^{-2n-1}}{q - q^{-1}}$$

is the $2n+1$ th quantum integer with parameter q [Jon83, Section 5.1].

We claim that

$$(1) \quad \langle \phi_t(g_n), g_n \rangle = [2n + 1]_\omega, \text{ if } n \geq 1,$$

